

# A NON-AUTONOMOUS MODEL PROBLEM FOR THE OSEEN-NAVIER-STOKES FLOW WITH ROTATING EFFECTS

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**ABSTRACT.** Consider the Navier-Stokes flow past a rotating obstacle with a general time-dependent angular velocity and a time-dependent outflow condition at infinity. After rewriting the problem on a fixed domain, one obtains a non-autonomous system of equations with unbounded drift terms. It is shown that the solution to a model problem in the whole space case  $\mathbb{R}^d$  is governed by a strongly continuous evolution system on  $L_\sigma^p(\mathbb{R}^d)$  for  $1 < p < \infty$ . The strategy is to derive a representation formula, similar to the one known in the case of non-autonomous Ornstein-Uhlenbeck equations. This explicit formula allows to prove  $L^p$ - $L^q$  estimates and gradient estimates for the evolution system. These results are key ingredients to obtain (local) mild solutions to the full nonlinear problem by a version of Kato's iteration scheme.

## 1. INTRODUCTION AND MAIN RESULT

In this paper we consider a model problem in  $\mathbb{R}^d$  for the flow of an incompressible, viscous fluid past a rotating obstacle with an additional time-dependent outflow condition at infinity. The equations describing this problem are the Navier-Stokes equations in an exterior domain varying in time with an additional condition for the velocity field at infinity.

In order to motivate our model problem, let  $\mathcal{O} \subset \mathbb{R}^d$  be a compact obstacle with smooth boundary, let  $\Omega := \mathbb{R}^d \setminus \mathcal{O}$  be the exterior of the obstacle and let  $m \in C([0, \infty); \mathbb{R}^{d \times d})$  be a continuous matrix-valued function. Then, the exterior of the rotated obstacle at time  $t > 0$  is represented by  $\Omega(t) := Q(t)\Omega$  where  $Q(t)$  solves the ordinary differential equation

$$\begin{cases} \partial_t Q(t) &= m(t)Q(t), \quad t > 0, \\ Q(0) &= \text{Id}. \end{cases} \quad (1.1)$$

With a prescribed velocity field  $v_\infty \in C^1([0, \infty); \mathbb{R}^d)$  at infinity, the equations for the fluid on the time-dependent domain  $\Omega(t)$  with no-slip boundary condition take the form

$$\begin{aligned} v_t - \Delta v + v \cdot \nabla v + \nabla q &= 0 && \text{in } \Omega(t) \times (0, \infty), \\ \operatorname{div} v &= 0 && \text{in } \Omega(t) \times (0, \infty), \\ v(t, y) &= m(t)y && \text{on } \partial\Omega(t) \times (0, \infty), \\ \lim_{|y| \rightarrow \infty} v(t, y) &= v_\infty(t) && \text{for } t \in (0, \infty), \\ v(0, y) &= u_0(y) && \text{in } \Omega, \end{aligned} \quad (1.2)$$

where  $v$  and  $q$  are the unknown velocity field and the pressure of the fluid, respectively.

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The disadvantage of this description is the variability of the domain  $\Omega(t)$ , and the fact that the equations do not fit into the  $L^p$ -setting, due the velocity condition at infinity. Assume for the time being that  $m(t)$  is skew symmetric for  $t > 0$ ; this implies that for all  $t > 0$  the matrix  $Q(t)$  is orthogonal. Then, by setting

$$x = Q(t)^T y, \quad u(t, x) = Q(t)^T (v(t, y) - v_\infty(t)), \quad p(t, x) = q(t, y), \quad (1.3)$$

the above equations can be transformed to the reference domain  $\Omega$  and the new velocity field  $u$  vanishes at infinity. Then (1.2) is equivalent to the following system of equations

$$\left. \begin{aligned} u_t - \Delta u - \mathcal{M}(t)x \cdot \nabla u + \mathcal{M}(t)u \\ + Q(t)^T v_\infty(t) \cdot \nabla u - Q(t)^T \partial_t v_\infty(t) \\ + u \cdot \nabla u + \nabla p \end{aligned} \right\} = 0 \quad \text{in } \Omega \times (0, \infty),$$

$$\begin{aligned} \operatorname{div} u &= 0 & \text{in } \Omega \times (0, \infty), \\ u(t, x) &= \mathcal{M}(t)x - Q(t)^T v_\infty(t) & \text{on } \partial\Omega \times (0, \infty), \\ \lim_{|x| \rightarrow \infty} u(t, x) &= 0 & \text{for } t \in (0, \infty), \\ u(0, x) &= u_0(x) & \text{in } \Omega, \end{aligned} \quad (1.4)$$

where  $\mathcal{M}(t) := Q(t)^T m(t) Q(t)$ . The main difficulty in dealing with this problem arises since the term  $\mathcal{M}(t)x \cdot \nabla$  has unbounded coefficients. In particular, the lower order terms cannot be treated by classical perturbation theory for the Stokes operator.

Note that even if we assume that  $m(t) \equiv m$  is independent of time (this implies that also  $\mathcal{M}(t) \equiv \mathcal{M}$  is independent of time), equation (1.4) is still non-autonomous due to the time-dependent first order term  $Q(t)^T v_\infty \cdot \nabla$  (except in some special cases discussed below).

However, by using localization techniques similar to [GHH06], this problem is finally reduced to a model problem in  $\mathbb{R}^d$  and a model problem in a bounded domain. Since  $Q(t)\partial_t v_\infty(t) \equiv F(t)$ ,  $t > 0$ , i.e. it is constant in space, we may put this term in the pressure  $p$ . Hence, in this paper we discuss the following linearized model problem in  $\mathbb{R}^d$

$$\begin{aligned} u_t - \Delta u - (M(t)x + f(t)) \cdot \nabla u + M(t)u + \nabla p &= 0 & \text{in } \mathbb{R}^d \times (0, \infty), \\ \operatorname{div} u &= 0 & \text{in } \mathbb{R}^d \times (0, \infty), \\ u(0) &= u_0 & \text{in } \mathbb{R}^d, \end{aligned} \quad (1.5)$$

where we allow general coefficients  $M \in C([0, \infty); \mathbb{R}^{d \times d})$  and  $f \in C([0, \infty); \mathbb{R}^d)$ . If we set  $M(t) := Q(t)^T m(t) Q(t)$  and  $f(t) := -Q(t)^T v_\infty(t)$  then we obtain the linearization of equation (1.4) with  $\Omega = \mathbb{R}^d$ . Such a model problem also arises in the analysis of a rotating body with translational velocity  $-v_\infty(t)$ , see [Far05].

Existence and uniqueness of a mild solution of an autonomous variant of problem (1.2) *without* an outflow condition, i.e.  $v_\infty \equiv 0$ , and  $m(t) \equiv m$ , was investigated in quite a few papers, see [His99a], [His99b], [GHH06] and [HS05]. Hishida was even able to deal with a time dependent rotation in [His01], however only for angular velocities of a special form.

For the problem including an additional outflow condition at infinity, there are only a few results. Indeed, in the special case, where  $m(t)x = \omega(t) \times x$  and  $\omega : [0, \infty) \rightarrow \mathbb{R}^3$  is the angular velocity of the obstacle and  $v_\infty : [0, \infty) \rightarrow \mathbb{R}^3$  a time-dependent outflow velocity, Borchers [Bor92] constructed weak non-stationary solutions for the equations (1.4). Moreover, Shibata [Shi08] studied the special case where  $m(t) \equiv m$ ,  $v_\infty(t) = v_\infty$  and  $mv_\infty = 0$ . The

condition  $mv_\infty = 0$ , i.e.  $Q(t)^T v_\infty = kv_\infty$  for  $k \in \{-1, 1\}$ , ensures that (1.4) is still an autonomous equation and the solution of (1.4) is governed by a  $C_0$ -semigroup which is *not* analytic. The physical meaning of the additional condition  $mv_\infty = 0$  is that the outflow direction of the fluid is parallel to the axis of rotation of the obstacle. The stationary problem of this latter situation was analysed in [Far05].

The assumption  $mv_\infty = 0$  was recently relaxed by the second author in [Han10]. Indeed, he was able to deal with the model problem in  $\mathbb{R}^d$  where  $m(t)v_\infty \neq 0$  and  $v_\infty(t) \equiv v_\infty$ . However he assumes that  $m(t)$  and  $m(s)$  commute for all  $t, s > 0$  which can physically be interpreted by the fact that the axis of rotation is fixed.

The aim of this work is to remove the latter additional condition, i.e.  $m(t)$  and  $m(s)$  need not to commute and  $v_\infty$  may be time-dependent.

As usual the Helmholtz projection  $\mathbb{P}$  allows us to rewrite (1.5) as an abstract Cauchy problem in  $L_\sigma^p(\mathbb{R}^d)$ , where  $L_\sigma^p(\mathbb{R}^d)$  denotes the space of all solenoidal vector fields in  $L^p(\mathbb{R}^d)^d$ :

$$\begin{aligned} u'(t) - A(t)u(t) &= 0, \quad t > 0, \\ u(0) &= u_0. \end{aligned} \tag{1.6}$$

Here:

$$\begin{aligned} A(t)u &:= \mathbb{P}(\Delta u + (M(t)x + f(t)) \cdot \nabla u + M(t)u) \\ D(A(t)) &:= \{u \in W^{2,p}(\mathbb{R}^d)^d \cap L_\sigma^p(\mathbb{R}^d) : M(t)x \cdot \nabla u \in L^p(\mathbb{R}^d)^d\}. \end{aligned}$$

Note that it immediately follows from [HS05] that for fixed  $t > 0$ , the operator  $A(t)$  is the generator of a  $C_0$ -semigroup, which is not analytic. The fact that the semigroup is not analytic prevents us from employing standard generation results for evolution systems, see [Paz83, Chapter 5] and references therein. For the same reason,  $L^p$ - $L^q$  estimates and gradient estimates don't follow from standard arguments.

Therefore, we first derive a representation formula for the solution of (1.5). In order to derive this representation formula we transform (1.5) to a non-autonomous heat equation which can be explicitly solved, see Section 3. It turns out that the transformation to a non-autonomous heat equation is crucial to deal with our problem in this generality since the different transformation used in [Han10] caused the additional assumption that  $M(t)$  and  $M(s)$  commute for all  $t, s > 0$ .

In the following we denote by  $\{U(t, s)\}_{t,s \geq 0}$  the evolution system on  $\mathbb{R}^d$  generated by the family of matrices  $\{-M(t)\}_{t \geq 0}$ , i.e.

$$\begin{cases} \partial_t U(t, s) &= -M(t)U(t, s), \\ U(s, s) &= \text{Id}. \end{cases} \tag{1.7}$$

Note that  $\partial_s U(t, s) = U(t, s)M(s)$ .

We are now ready to present our main result.

**Theorem 1.1.** *Let  $1 < p < \infty$ ,  $M \in C([0, \infty); \mathbb{R}^{d \times d})$  and  $f \in C([0, \infty); \mathbb{R}^d)$ . The the solution of (1.6) is governed by a strongly continuous evolution system  $\{T(t, s)\}_{t \geq s \geq 0} \subset \mathcal{L}(L_\sigma^p(\mathbb{R}^d)^d)$ . Moreover, the evolution system  $\{T(t, s)\}_{t \geq s \geq 0}$  admits the following properties:*

(a) For  $T_0 > 0$  set  $M_{T_0} := \sup\{\|U(t, s)\| : t, s \in [0, T_0]\}$ . Then for  $1 < p < \infty$  and  $p \leq q \leq \infty$  there exists  $C := C(M_{T_0}, d) > 0$  such that for  $u \in L_\sigma^p(\mathbb{R}^d)$

$$\|T(t, s)u\|_{L_\sigma^q(\mathbb{R}^d)} \leq C(t-s)^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|u\|_{L_\sigma^p(\mathbb{R}^d)}, \quad 0 \leq s < t < T_0, \quad (1.8)$$

$$\|\nabla T(t, s)u\|_{L^q(\mathbb{R}^d)} \leq C(t-s)^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}} \|u\|_{L_\sigma^p(\mathbb{R}^d)}, \quad 0 \leq s < t < T_0. \quad (1.9)$$

In particular, if the evolution system  $\{U(t, s)\}_{s, t \geq 0}$  is uniformly bounded, i.e.  $M_{T_0} \leq M$ , for some  $M > 0$  and all  $T_0 > 0$ , we may set  $T_0 = \infty$ .

(b) For  $1 < p < q < \infty$ ,  $s \geq 0$  and  $u \in L_\sigma^p(\mathbb{R}^d)$  we have

$$\lim_{t \rightarrow s, t > s} (t-s)^{\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|T(t, s)u\|_{L_\sigma^q(\mathbb{R}^d)} = 0 \text{ and } \lim_{t \rightarrow s, t > s} (t-s)^{\frac{1}{2}} \|\nabla T(t, s)u\|_{L^p(\mathbb{R}^d)} = 0.$$

Next we consider the nonlinear problem

$$\begin{aligned} u'(t) - A(t)u(t) + \mathbb{P}((u(t) \cdot \nabla)u(t)) &= 0, \quad t > 0, \\ u(0) &= u_0, \end{aligned} \quad (1.10)$$

with initial value  $u_0 \in L_\sigma^p(\mathbb{R}^d)$ .

For given  $0 < T_0 \leq \infty$ , we call a function  $u \in C([0, T_0]; L_\sigma^p(\mathbb{R}^d))$  a *mild solution* of (1.10) if  $u$  satisfies the integral equation

$$u(t) = T(t, 0)u_0 - \int_0^t T(t, s)\mathbb{P}((u(s) \cdot \nabla)u(s))ds, \quad t > 0, \quad (1.11)$$

in  $L_\sigma^p(\mathbb{R}^d)$ . By adjusting Kato's iteration scheme (see [Kat84]) to our situation the existence of a unique (local) mild solution follows, cf. [Han10] for details.

**Corollary 1.2.** *Let  $2 \leq d \leq p \leq q < \infty$ ,  $M \in C([0, \infty); \mathbb{R}^{d \times d})$ ,  $f \in C([0, \infty); \mathbb{R}^d)$  and  $u_0 \in L_\sigma^p(\mathbb{R}^d)$ . Then there exists  $T_0 > 0$  and a unique mild solution  $u \in C([0, T_0]; L_\sigma^p(\mathbb{R}^d))$  of (1.10), which has the properties*

$$t^{\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} u(t) \in C([0, T_0]; L_\sigma^q(\mathbb{R}^d)), \quad (1.12)$$

$$t^{\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)+\frac{1}{2}} \nabla u(t) \in C([0, T_0]; L^q(\mathbb{R}^d)^{d \times d}). \quad (1.13)$$

If  $p < q$ , then in addition

$$t^{\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|u(t)\|_{L^q(\mathbb{R}^d)} + t^{\frac{1}{2}} \|\nabla u(t)\|_{L^p(\mathbb{R}^d)} \rightarrow 0 \quad \text{as } t \rightarrow 0. \quad (1.14)$$

Moreover, in the case  $d = p$  we may set  $T_0 = +\infty$  provided  $\|u_0\|_{L^d(\mathbb{R}^d)}$  is small enough and  $\{U(t, s)\}_{s, t \geq 0}$  is uniformly bounded.

**Remark 1.3.** In particular,  $\{U(t, s)\}_{s, t \geq 0}$  is uniformly bounded if  $M(t)$  is skew symmetric for all  $t > 0$ .

## 2. PROOF OF THEOREM 1.1

Let  $M$  be as in Theorem 1.1, and let  $\{U(t, s)\}_{s, t \geq 0}$  be the evolution system on  $\mathbb{R}^d$  that satisfies (1.7). We consider the system of parabolic equations of the form

$$\begin{cases} \partial_t u(t, x) - \mathcal{A}(t)u(t, x) &= 0, & t > s, x \in \mathbb{R}^d, \\ u(s, x) &= \varphi(x), & x \in \mathbb{R}^d, \end{cases} \quad (2.1)$$

for  $s \geq 0$  fixed, initial value  $\varphi \in L^p(\mathbb{R}^d)^d$  and some  $p \in (1, \infty)$ . Here the family of operators  $\mathcal{A}(t)$  is of the form

$$\mathcal{A}(t)u(x) := \left( \Delta u_i(t, x) + \langle M(t)x + f(t), \nabla u_i(t, x) \rangle \right)_{i=1}^d - M(t)u(t, x), \quad t > 0, x \in \mathbb{R}^d.$$

As in [GL08, Lemma 3.2] or [Han10], we first develop an explicit representation formula. To be more precise, we show in Section 3 that for  $p \in (1, \infty)$  and  $\varphi \in L^p(\mathbb{R}^d)^d$  the solution  $u$  to (2.1) is governed by a strongly continuous evolution system  $\{\tilde{T}(t, s)\}_{t \geq s} \subset \mathcal{L}(L^p(\mathbb{R}^d)^d)$  which is explicitly given by

$$u(t, x) := (\tilde{T}(t, s)\varphi)(x) := (k(t, s, \cdot) * \varphi)(U(s, t)x + g(t, s)), \quad t > s, x \in \mathbb{R}^d, \quad (2.2)$$

where

$$\begin{aligned} k(t, s, x) &:= \frac{1}{(4\pi)^{d/2}(\det Q_{t,s})^{1/2}} U(t, s) e^{-\frac{1}{4}\langle Q_{t,s}^{-1}x, x \rangle} dy, \quad t > s \geq 0, x \in \mathbb{R}^d, \\ g(t, s) &:= \int_s^t U(s, r)f(r)dr, \quad Q_{t,s} := \int_s^t U(s, r)U^*(s, r)dr, \quad t \geq s \geq 0. \end{aligned} \quad (2.3)$$

Similar to [DPL07] one can show that for  $\varphi \in C_c^\infty(\mathbb{R}^d)^d$  the solution  $u$  of (2.1) given by (2.2) is a classical solution.

A simple calculation shows that  $\operatorname{div} \tilde{T}(t, s)\varphi = 0$  for  $\varphi \in C_{c,\sigma}^\infty(\mathbb{R}^d)$  and  $t \geq s \geq 0$ . Hence, the restriction  $T(t, s) := \tilde{T}(t, s)|_{L_\sigma^p(\mathbb{R}^d)}$  is an evolution system on  $L_\sigma^p(\mathbb{R}^d)$ . In particular,  $u(t) := T(t, 0)u_0$  is a solution to (1.6).

By similar arguments as in the proofs of [GL08, Lemma 3.2] or [Han10, Lemma 2.4], for  $T_0 > 0$  there exists  $C := C(d, M_{T_0}) > 0$  (see Theorem 1.1 for the definition of  $M_{T_0}$ ) such that

$$\begin{aligned} \|Q_{t,s}^{-\frac{1}{2}}\| &\leq C(t-s)^{-\frac{1}{2}}, \quad 0 \leq s < t < T_0, \\ (\det Q_{t,s})^{\frac{1}{2}} &\geq C(t-s)^{\frac{d}{2}}, \quad 0 \leq s < t < T_0. \end{aligned} \quad (2.4)$$

Moreover, if  $M_{T_0}$  is uniformly bounded in  $T_0$  we may write  $T_0 = \infty$  in (2.4).

*Proof of Theorem 1.1.* We start by showing the estimate (1.8). Let  $T_0 > 0$ . By the change of variables  $\xi = U(s, t)x$  and by Young's inequality we obtain

$$\|T(t, s)u\|_{L_\sigma^q(\mathbb{R}^d)} \leq |\det U(s, t)|^{\frac{1}{q}} \|k(t, s, \cdot)\|_{L^r(\mathbb{R}^d)} \|u\|_{L_\sigma^p(\mathbb{R}^d)}, \quad t > s \geq 0,$$

where  $1 < r < \infty$  with  $\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}$ . Further, by the change of variable  $y = Q_{t,s}^{1/2}z$  we obtain

$$\begin{aligned} \|k(t, s, \cdot)\|_{L^r(\mathbb{R}^d)}^r &= \|U(t, s)\| \int_{\mathbb{R}^d} \left( \frac{1}{(4\pi)^{\frac{d}{2}}} e^{-\frac{|z|^2}{4}} \right)^r (\det Q_{t,s})^{\frac{1-r}{2}} dz \\ &\leq C \|U(t, s)\| (\det Q_{t,s})^{\frac{1-r}{2}}, \quad t \geq s \geq 0, \end{aligned}$$

for some  $C > 0$ . Now (2.4) yields (1.8).

To prove the gradient estimate (1.9), we first observe that

$$\nabla T(t, s)u(x) = \int_{\mathbb{R}^d} u(U(s, t)x + g(t, s)k(t, s, y)) (U^T(s, t)Q_{s,t}^{-1}y)^T dy, \quad t > s \geq 0, x \in \mathbb{R}^d.$$

Now, (1.9) follows similarly as above.

Since (2.1) is uniquely solvable for  $\varphi \in C_c^\infty(\mathbb{R}^d)^d$ , see Section 3, the law of evolution is valid, i.e.

$$\tilde{T}(t, s)\varphi = \tilde{T}(t, r)\tilde{T}(r, s)\varphi, \quad (2.5)$$

holds for  $0 \leq s \leq r \leq t$  and every  $\varphi \in C_c^\infty(\mathbb{R}^d)^d$ . The density of  $C_c^\infty(\mathbb{R}^d)^d$  in  $L^p(\mathbb{R}^d)^d$  yields that (2.5) even holds for all  $\varphi \in L^p(\mathbb{R}^d)^d$ .

In order to prove the strong continuity of the map  $(t, s) \mapsto \tilde{T}(t, s)$  on  $0 \leq s \leq t$  we apply the change of the variables  $y = Q_{t,s}^{1/2}z$ , to see that

$$\tilde{T}(t, s)\varphi(x) = \frac{1}{(4\pi)^{\frac{d}{2}}} U(t, s) \cdot \int_{\mathbb{R}^d} \varphi(U(s, t)x + g(t, s) - Q_{t,s}^{\frac{1}{2}}z) e^{-\frac{|z|^2}{4}} dz$$

holds. For  $t > s$  fixed, we pick two sequences  $(t_n)_{n \in \mathbb{N}}$  and  $(s_n)_{n \in \mathbb{N}}$  such that  $t_n \geq s_n$  holds for every  $n \in \mathbb{N}$  and  $(t_n, s_n) \rightarrow (t, s)$  as  $n \rightarrow \infty$ . For every  $\varphi \in C_c^\infty(\mathbb{R}^d)^d$  and every  $x \in \mathbb{R}^d$  we now obtain

$$\varphi(U(s_n, t_n)x + g(t_n, s_n) - Q_{t_n, s_n}^{\frac{1}{2}}z) \rightarrow \varphi(U(s, t)x + g(t, s) - Q_{t,s}^{\frac{1}{2}}z)$$

as  $n \rightarrow \infty$ . Lebesgue's theorem now yields  $\tilde{T}(t_n, s_n)\varphi \rightarrow \tilde{T}(t, s)\varphi$  as  $n \rightarrow \infty$  for every  $\varphi \in C_c^\infty(\mathbb{R}^d)^d$ . The density of  $C_c^\infty(\mathbb{R}^d)^d$  in  $L^p(\mathbb{R}^d)^d$  implies the strong continuity.

In order to prove Theorem 1.1(b) let  $u \in L_\sigma^p(\mathbb{R}^d)$ ,  $t - s \leq 1$  and choose  $(u_n)_{n \in \mathbb{N}} \subset C_{c,\sigma}^\infty(\mathbb{R}^d) \subset L_\sigma^p(\mathbb{R}^d)$ , such that  $\lim_{n \rightarrow \infty} \|u - u_n\|_{L^p(\mathbb{R}^d)} = 0$ . The triangle inequality together with the  $L^p$ - $L^q$  estimates (1.8) imply that there exist constants  $C_1, C_2 > 0$  such that

$$\begin{aligned} & (t - s)^{\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} \|T(t, s)u\|_{L_\sigma^q(\mathbb{R}^d)} \\ & \leq (t - s)^{\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} \|T(t, s)(u - u_n)\|_{L_\sigma^q(\mathbb{R}^d)} + (t - s)^{\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} \|T(t, s)u_n\|_{L_\sigma^q(\mathbb{R}^d)} \\ & \leq C_1 \|u - u_n\|_{L_\sigma^p(\mathbb{R}^d)} + C_2 (t - s)^{\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} \|u_n\|_{L_\sigma^q(\mathbb{R}^d)}, \quad 0 \leq t - s \leq 1, \quad n \in \mathbb{N}. \end{aligned}$$

Hence,  $\lim_{t \rightarrow s} (t - s)^{\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} \|T(t, s)u\|_{L_\sigma^q(\mathbb{R}^d)} = 0$  by letting first  $t \rightarrow s$  and then  $n \rightarrow \infty$ . The second assertion in Theorem 1.1(b) is proved in a similar way.  $\square$

### 3. REPRESENTATION FORMULA

In this section the representation formula (2.2) is derived. The general idea is to do a coordinate transformation in order to eliminate the unbounded drift and the zero order term of the operator  $\mathcal{A}(t)$ . For this purpose we set

$$z := U(s, t)x + g(t, s),$$

where

$$g(t, s) := \int_s^t U(s, r)f(r)dr,$$

and we look for a solution  $u$  of (2.1) with initial value  $\varphi \in C_c^\infty(\mathbb{R}^d)^d$  in the form

$$u(t, x) = U(t, s)w(t, U(s, t)x + g(t, s)). \quad (3.1)$$

By recalling (1.7) we obtain from a straightforward computation that

$$\begin{aligned} \partial_t u(t, x) = & -M(t)U(t, s)w(t, z) + U(t, s) \left( \langle U(s, t)M(t)x + U(s, t)f(t), \nabla w_i(t, z) \rangle \right)_{i=1}^d \\ & + U(t, s)\partial_t w(t, z), \end{aligned}$$

holds. Moreover, we can write equation (3.1) component-wise as

$$u_i(t, x) = \sum_{j=1}^d U_{ij}(t, s)w_j(t, U(s, t)x + g(t, s)), \quad \text{for } i = 1, \dots, d,$$

and thus for the spatial derivatives of  $u$  we obtain

$$\begin{aligned} \nabla u_i(t, x) &= \sum_{j=1}^d U_{ij}(t, s)U^*(s, t)\nabla w_j(t, z), \\ \nabla^2 u_i(t, x) &= \sum_{j=1}^d U_{ij}(t, s)U^*(s, t)\nabla^2 w_j(t, z)U(s, t). \end{aligned}$$

In particular, the drift term can be written as

$$\langle M(t)x + f(t), \nabla u_i(t, x) \rangle = \sum_{j=1}^d U_{ij}(t, s) \langle U(s, t)M(t)x + U(s, t)f(t), \nabla w_j(t, z) \rangle.$$

Thus, the function  $u$  solves problem (2.1) if and only if for every  $i = 1, \dots, d$ , the function  $w_i : \mathbb{R}^d \rightarrow \mathbb{R}$  is a solution to

$$\begin{cases} \partial_t w_i(t, z) &= \text{Tr}[U(s, t)U^*(s, t)\nabla^2 w_i(t, z)], & t > s, z \in \mathbb{R}^d, \\ w_i(s, z) &= \varphi_i(z), & z \in \mathbb{R}^d. \end{cases} \quad (3.2)$$

By our transformation we now obtained an uncoupled system of parabolic equations with coefficients only depending on  $t$ . More precisely, for  $i = 1, \dots, d$ , the equation (3.2) is a non-autonomous heat equation. It is well known that such a problem can be uniquely solved (cf. [DPL07, Proposition 2.1]) and that for every  $\varphi_i \in C_c^\infty(\mathbb{R}^d)$  its unique solution is explicitly given by the formula

$$w_i(t, z) = \frac{1}{(4\pi)^{\frac{d}{2}}(\det Q_{t,s})^{\frac{1}{2}}} \int_{\mathbb{R}^d} \varphi_i(z - y) e^{-\frac{1}{4}\langle Q_{t,s}^{-1}y, y \rangle} dy, \quad (3.3)$$

where

$$Q_{t,s} = \int_s^t U(s, r)U^*(s, r)dr. \quad (3.4)$$

Now, via (3.1), the unique solution to our original problem (2.1) is given by the representation formula

$$u(t, x) = (k(t, s, \cdot) * u)(U(s, t)x + g(t, s)), \quad (3.5)$$

where the kernel  $k(t, s, x)$  is defined in (2.3).

Note that the right hand side of (3.5) is even well defined for each  $L^p(\mathbb{R}^d)^d$ -function  $\varphi$ . Thus, this explicit formula can be used to define an evolution system on  $L^p(\mathbb{R}^d)^d$  in the following

way. For  $\varphi \in L^p(\mathbb{R}^d)^d$  we set

$$\tilde{T}(t, s)\varphi := \begin{cases} \varphi & \text{for } t = s, \\ (k(t, s, x) * \varphi)(U(s, t)x + g(t, s)) & \text{for } t > s. \end{cases}$$

Since problem (3.2) is uniquely solvable it follows via (3.1) that  $\tilde{T}(t, s)\varphi$  is the unique solution of (2.1) for initial value  $\varphi \in C_c^\infty(\mathbb{R}^d)^d$ .

## REFERENCES

- [Bor92] W. Borchers, *Zur Stabilität und Faktorisierungsmethode für die Navier-Stokes-Gleichungen inkompressibler viskoser Flüssigkeiten*, Habilitation, 1992, Habilitation, Universität Paderborn.
- [DPL07] G. Da Prato and A. Lunardi, *Ornstein-Uhlenbeck operators with time periodic coefficients*, J. Evol. Equ. **7** (2007), 587–614.
- [Far05] R. Farwig, *An  $L^q$ -analysis of viscous fluid flow past a rotating obstacle*, Tohoku Math. J. (2) **58** (2005), 129–147.
- [GHH06] M. Geissert, H. Heck, and M. Hieber,  *$L^p$ -theory of the Navier-Stokes flow in the exterior of a moving or rotating obstacle*, J. Reine Angew. Math. **596** (2006), 45–62.
- [GL08] M. Geissert and A. Lunardi, *Invariant measures and maximal  $L^2$  regularity for nonautonomous Ornstein-Uhlenbeck equations*, J. Lond. Math. Soc. (2) **77** (2008), 719–740.
- [Han10] T. Hansel, *On the navier-stokes equations with rotating effect and prescribed outflow velocity*, Journal of Mathematical Fluid Mechanics, to appear.
- [His99a] T. Hishida, *An existence theorem for the Navier-Stokes flow in the exterior of a rotating obstacle*, Arch. Ration. Mech. Anal. **150** (1999), 307–348.
- [His99b] T. Hishida, *The Stokes operator with rotation effect in exterior domains*, Analysis (Munich) **19** (1999), 51–67.
- [His01] T. Hishida, *On the Navier-Stokes flow around a rigid body with a prescribed rotation*, Proceedings of the Third World Congress of Nonlinear Analysts, Part 6 (Catania, 2000), vol. 47, 2001, pp. 4217–4231.
- [HS05] M. Hieber and O. Sawada, *The Navier-Stokes equations in  $\mathbb{R}^n$  with linearly growing initial data*, Arch. Ration. Mech. Anal. **175** (2005), 269–285.
- [Kat84] T. Kato, *Strong  $L^p$ -solutions of the Navier-Stokes equation in  $\mathbf{R}^m$ , with applications to weak solutions*, Math. Z. **187** (1984), 471–480.
- [Paz83] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, 1983.
- [Shi08] Y. Shibata, *On the Oseen semigroup with rotating effect*, Functional analysis and evolution equations, Birkhäuser, Basel, 2008, pp. 595–611.

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